Stability Analysis of Pipe With Connectors in Horizontal Wells

Guohua Gao,* SPE, Qinfeng Di, Shanghai University; Stefan Miska, SPE, University of Tulsa; and Wenchang Wang, Shanghai University

Summary

Except for coiled tubing, most tubular goods used for downhole operations (such as drillpipe and sucker rod) have connectors. Because a connector and the pipe body have different outer radii, the deformation and buckling behavior of a pipe with connectors constrained in a wellbore is much more complicated. However, most buckling models were established by neglecting the existence and effects of connectors.

In this paper, buckling equations of a pipe with connectors in horizontal wells were derived with application of elastic-beam theory. The axis of an unbuckled pipe is a 2D curve in the vertical plane and has three configurations—no contact, point contact, and wrap contact. We derived the two critical distances between connectors, \( L_{c1} \) and \( L_{c2} \), beyond which a pipe changes its configuration from one to another. The authors proposed an algorithm to determine the critical force \( F_{crs} \) of buckling by numerically solving the buckling equations using the fourth-order Rounge-Kutta method.

Both the distance between two adjacent connectors \( (L_c) \) and the radius difference between a connector and the pipe body \( (\Delta r) \) have significant impact on the critical force, in addition to net deflection is negligible. \( F_{crs} \) increases as \( \Delta r \) increases. However, when \( L_c \) is close to \( L_{c1} \), effects of radial displacement become significant, and \( F_{crs} \) decreases dramatically as \( \Delta r \) increases. \( F_{crs} \) decreases as \( L_c \) increases when \( L_c < L_{c1} \), and it reaches its minimum at \( L_c = L_{c1} \). When \( L_c > L_{c1} \), \( F_{crs} \) fluctuates as \( L_c \) increases. Some curves of \( L_{c1} \), \( L_{c2} \), and \( F_{crs} \), all in dimensionless forms, were calculated and presented in this paper for practical applications.

Our numerical results show that the critical force may reduce by 20 to 60% for commonly used drillpipes and sucker rods with centralizers, which indicates that a pipe string designed without considering the effects of connectors may be risky. The results presented in this paper may provide some practical guidance for optimal design of centralizers for sucker-rod strings, or may avoid some risks because of improper design of drillpipe strings.

Introduction

Buckling of pipe or tubing in wellbores may cause serious downhole problems. For example, helically buckled drillpipe may be locked up and not be able to transfer required weight on bit (WOB) or torque to the bit. Many researchers (Lubinski 1950; Lubinski et al. 1962; Paslay and Bogy 1964; Dawson and Paslay 1984; Mitchell 1988, 1999; Chen et al. 1990; Kyllingstad, 1995; He et al. 1995; He and Kyllingstad 1995; Gao et al. 1994; Gao 1996; Suryanarayana and McCann, 1994, 1995; Delli et al. 1998; Qiu et al. 1998; Qiu 1999; Duman et al. 2003; Mitchell and Miska 2006; Gao and Miska 2009a, b; Menand et al. 2009) have made significant contributions to different aspects of pipe buckling in various wellbores. Mitchell (2008) and Cunha (2004) presented a detailed review of literature on this topic. Both theoretical analyses and experiments have shown that a pipe may buckle and transform from its initial configuration to a sinusoidal-wave-like configuration, or a helix.

Paslay and Bogy (1964) studied the stability and sinusoidal buckling of tubing constrained in an inclined borehole with application of energy method. Dawson and Paslay (1984) derived the formula of critical axial force of sinusoidal buckling,

\[
F_{crs} = 2 \frac{EI}{r_0} \sin \frac{\pi}{a} \text{............................... (1)}
\]

where \( a \) is the inclination angle of the wellbore.

Instead of applying the energy method, Mitchell (1988) applied the equilibrium-analysis method and derived a fourth-order nonlinear ordinary-differential equation that describes the post-buckling behavior of tubing in inclined wells:

\[
\frac{d^4\theta}{dx^4} - 6 \frac{d^2\theta}{dx^2} + \frac{d\theta}{dx} \left( \frac{F}{EI} \frac{d\theta}{dx} + \frac{w}{EI} \sin \theta \right) = 0 \text{................. (2)}
\]

Mitchell (1997, 1999, 2002) further analyzed the post-buckling behaviors. It is obvious that \( \theta = 0 \) is a trivial solution of Eq. 2. When \( F > F_{crs} \), \( \theta = 0 \) becomes unstable. For small \( \theta \), Eq. 2 can be linearized as

\[
\frac{d^4\theta}{dx^4} + 2 \frac{d\theta}{dx} \frac{d\theta}{dx} + \frac{\beta}{\alpha} \theta = 0 \text{....................... (3)}
\]

where \( \beta = \frac{F}{2} \sqrt{\frac{E}{EI \sin \alpha}} \) and \( \alpha = \left( \frac{w}{EI \sin \alpha} \right)^{\frac{1}{2}} \) are, respectively, dimensionless axial force and dimensionless distance. Critical load of sinusoidal buckling of a pipe constrained in a horizontal well \( (x = 0.5\pi) \) can be determined from the general solution of Eq. 3 with respect to different boundary conditions (Gao et al. 1994; Gao 1996). The effects of boundary conditions can be neglected for a long pipe, and the critical load of sinusoidal buckling approaches \( \beta_{crs} = 1 \), which is exactly the same as that predicted by Eq. 1. Gao (1996) presented a perturbation solution of the buckling equation (Eq. 2) and showed that the sinusoidal post-buckling configuration of a long pipe constrained in a horizontal well can be approximated by (Gao and Miska 2010),

\[
\theta = \frac{4\sqrt{\beta - 1}}{\sqrt{11}} \sin x \text{....................... (4)}
\]

Eq. 4 indicates that the wave length of a sinusoidally buckled pipe constrained in a horizontal wellbore is

\[
\frac{w}{EI \sin \alpha} \left( \frac{E}{W} \right)^{\frac{1}{2}} \pi = 2\pi \text{....................... (5)}
\]

The analytical solution of Eq. 5 is consistent with the results derived by Dawson and Paslay (1984) with application of the energy method.

* Now with Shell.

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All the preceding results were developed under the assumption that a pipe continuously contacts the wall of a wellbore. However, this assumption does not hold for the case of a pipe with connectors. In fact, part of the pipe does not contact the wall of a wellbore because the outer radius of a connector is larger than that of the pipe body. Only a few papers considered the effect of connectors on buckling models. Paslay and Cermocky (1991; Mitchell 2000, 2003a, b; Duman et al. 2003; Mitchell and Miska 2006). Paslay and Cermocky (1991) investigated deformation and stress of drillstring with connectors in curved wellbores with constant curvature. Their results showed that an unbuckled pipe with connectors in a curved wellbore may take three different configurations (no contact, point contact, and wrap contact) when it is subjected to axial compressive or tensile force. However, the weight of pipe was ignored in their model and they did not investigate the stability problem. Mitchell investigated buckling of a pipe with connectors to be constrained in vertical wells (Mitchell 2000), horizontal wells (Mitchell 2003a), and curved wells (Mitchell 2003b). One assumption Mitchell used to derive the critical load of sinusoidal buckling and post-buckling behavior (Mitchell 2003a, b) is that the wave length of a sinusoidally buckled pipe is twice the length between two adjacent connectors. However, this assumption is not consistent with Eq. 5 when the outer radius of a connector equals that of the pipe body. As shown in Eq. 5, the wave length should also depend on bending stiffness (EI), weight (w), and the clearance between a pipe and the wellbore (rc), in addition to distance between two connectors (Lc) and the difference of outer radius (Ar) between a connector and the pipe body. Duman et al. (2003) conducted some experimental studies on the effect of tool joints on contact force and axial-force transfer in horizontal wellbores. The effect of connector or tool joint on critical load of buckling and post-buckling behavior has been neither well investigated nor well understood yet.

In this paper, we derived a buckling model for a pipe with connectors to be constrained in horizontal wells with application of equilibrium equations of an elastic beam. The buckling model is composed of two nonlinear differential equations with two basic unknowns—angular displacement (θ(x)) and radial displacement (r(x)). The configuration of a pipe’s axis is completely determined by the two basic unknowns. The two unknowns can be solved from the two nonlinear differential equations under proper boundary conditions.

When the axial compressive force (F) is smaller than the critical force of sinusoidal buckling (Fcr), the axis of a pipe is a 2D curve in the vertical plane. The trivial solution of the buckling equations, θ(x) = 0, represents the stable configuration of an unbuckled pipe. In case of a pipe without connectors, r(x) = r0 is a constant and indicates a continuous contact between the pipe and the wall of the wellbore. But the radial displacement r(x) of a pipe with connectors is no longer a constant. r(x) can be solved analytically from a linear differential equation that is simplified from the buckling equations by setting θ(x) = 0. From the general solution and boundary/contact conditions specified at each connector or contact point, we showed that the axis of an unbuckled pipe may take three different configurations—no contact, point contact, and linear (or wrap) contact—which is similar to the pipe body. Material properties include the weight of the pipe per unit length (w) and the bending stiffness of the pipe (EI).

In case of point contact (as shown in Fig. 1b), the middle point of a pipe between two adjacent connectors contacts the wall of the wellbore, and a normal concentrated contact force Nc acts on the ith connector at xi = Lci; here, we assume the first connector is located at the dead end (xi = 0).

In case of point contact (as shown in Fig. 1b), the middle point of a pipe between two adjacent connectors contacts the wall of the wellbore, and a normal concentrated contact force Nc acts on the pipe at the middle point [xi,m = (i + 0.5)L]. In case of wrap contact, in addition to normal contact force Nc(x) that is linearly distributed over the interval of [xi,1 ≡ x1; xi,2 ≡ x2], we showed that r(x) can be solved analytically from a linear differential equation that is simplified from the buckling equations by setting θ(x) = 0. From the general solution and boundary/contact conditions specified at each connector or contact point, we showed that the axis of an unbuckled pipe may take three different configurations—no contact, point contact, and linear (or wrap) contact—which is similar to the results obtained by Paslay and Cermocky (1991). We presented a detailed analysis of these three different configurations, and derived the two critical distances between two connectors, Lc,1 and Lc,2. An unbuckled pipe will change its configuration from no contact to point contact when Lc > Lc,1 or from point contact to wrap contact when Lc > Lc,2. Contact force acting at a connector and contact force acting at a contact point were also derived.

When F > Fcr, the pipe’s axis will change from a 2D curve to a 3D snaking curve [i.e., θ(x) ≠ 0]. An important issue is how to determine the critical force of a pipe with connectors. Although the effect of friction on stability and post-buckling behavior of a pipe constrained in a wellbore is significant (Gao and Miska 2009a, b; Suryanarayana and McCann 1994, 1995), we did not consider the effect of friction in this paper for simplicity. As shown by Eq. 4, θ(x) becomes very small when F is close enough to Fcr, and thus, the buckling equations can be linearized by neglecting higher terms of θ(x). The linearized buckling equation with respect to radial displacement is the same as the equation for an unbuckled pipe, and its solutions with three different configurations can be applied directly to the other linearized buckling equation with respect to angular displacement. For small θ(x), the nonlinear buckling equation with respect to angular displacement becomes a linear differential equation with coefficients being determined by the radial displacement that has been solved from the linear differential equation of an unbuckled pipe.

However, analytical solution for the linearized buckling equation of angular displacement is still not available. We proposed a procedure to obtain the critical force by numerically solving the linearized buckling equation with application of the fourth order Ronge-Kutta method.

### Buckling Equations

#### Basic Assumptions.

1. Radius of the wellbore is constant.
2. The clearance between the pipe and the wellbore is so small that the deformation of pipe is elastic.
3. A connector is regarded as a rigid disk with zero length.
4. No torque and no friction.

#### Geometric Parameters, Material Properties, and External Forces.

An unbuckled pipe with connectors constrained in a horizontal well may take three different equilibrium configurations: no contact (Fig. 1a), point contact (Fig. 1b), and wrap contact (Fig. 1c). The equilibrium configuration of a pipe’s axis depends on the following geometric parameters, material properties, and external forces acting on the pipe.

Geometric parameters include the inner radius of the wellbore (ri), the outer radius of a connector (ro), the outer radius of the pipe body (rp), the distance between two adjacent connectors (Lc), and total length of the pipe (L). r0 = r0 − rp0 denotes the clearance between the pipe and the wellbore, and Ar = ri − rp0 denotes the difference of outer radius between a connector and the pipe body. Material properties include the weight of the pipe (w) and the bending stiffness of the pipe (EI).

### Dimensionless Parameters and Dimensionless Variables.

Let

\[
\mu = \left(\frac{\mu}{\mu_r}\right)^{0.25},
\]

and we define dimensionless distances \(\xi = \mu x\), \(\zeta_0 = \mu L_0\), \(\zeta_1 = \mu L_1\), and \(\zeta_2 = \mu L_2\). \(\gamma = \frac{\theta}{\theta_0}\) denotes dimensionless radial displacement, and \(\Delta = \frac{\theta}{\theta_0}\) is dimensionless radius difference. Dimensionless axial force is defined as \(\beta = \frac{F}{F_{cr}}\). Dimensionless concentrated normal contact forces are defined as

\[n_i = \frac{N_{ci}}{N_{c0}}, n_p = \frac{N_{cp}}{N_{c0}}, \text{ and } n_w = \frac{N_{cw}}{N_{c0}}. \]

\(n\) represents dimensionless linearly distributed normal contact force.
Static-Buckling Equations. A detailed analysis of elastic deformation of a pipe constrained in an inclined well is presented in Appendix A. Buckling equations are derived with application of static equilibrium equations in Appendix B. Using the dimensionless parameters and variables just defined, and assuming \( \theta = 0.5\pi \) (a horizontal well), the buckling equations (Eqs. B-16 and B-17) can be normalized as

\[
\frac{n}{C_0} \left\{ \frac{1}{C_1} \frac{d^4 \gamma}{d z^4} - 2 \left( \frac{d \theta}{d z} \right)^3 + \frac{2 \beta}{d z} \left( \frac{d \theta}{d z} \right)^2 \right\} + \sin \theta = 0.
\]

\[
\frac{n}{C_0} \left\{ \frac{1}{C_1} \frac{d^4 \gamma}{d z^4} - 2 \left( \frac{d \theta}{d z} \right)^3 + \frac{2 \beta}{d z} \left( \frac{d \theta}{d z} \right)^2 \right\} + \sin \theta = 0.
\]

As shown by Eqs. 6 and 7, the radial displacement \( \gamma(z) \) is nonlinearly coupled with the angular displacement \( \theta(z) \). Theoretically, we can solve both \( \theta(z) \) and \( \gamma(z) \) from Eqs. 6 and 7 under appropriate boundary conditions. However, it is very difficult to obtain a general solution of the buckling equations. Some specific solutions may provide insights for further investigation. For example, the solutions of an unbuckled pipe with respect to \( \theta(z) = 0 \) can

**Fig. 1—Three types of equilibrium configurations of an unbuckled pipe with connectors in horizontal wells.**
provide an accurate enough approximation of the radial displacement for stability analysis. In the following sections, we will investigate the static equilibrium configurations of an unbuckled pipe with connectors. If \( \theta(z) \neq 0 \) holds for a buckled pipe. When the axial force is close enough to the critical force of buckling, \( \theta(z) \) is very small, and thus Eqs. 6 and 7 can be linearized as

\[
\frac{d^2\gamma}{dz^2} + \frac{d^2\theta}{dz^2} - 2\beta \frac{d\gamma}{dz} = 0
\]

where \( A_1 \) and \( A_2 \) are integral constants that can be determined by proper boundary conditions at \( z_{i+1} \) (in case of no contact), the contact conditions at the middle point \( z_m = z_i + 0.5z_e \) (in case of point contact), or at \( z_d = z_i + z_e \) (in case of wrap contact). Solutions of Eqs. 11 and 12 indicate that the radial displacement for the part of pipe in the \( i \)th segment between the two adjacent connectors at \( z_i \) and \( z_{i+1} \) is exactly the same as that in other segments. Without lost generality, we will discuss only the solutions in one segment.

Given \( \Delta \gamma_c, \eta, \) and \( \beta, \) we can determine the pipe’s configuration using Eqs. 11 and 12 under appropriate boundary conditions. Plots with different styles in Fig. 2 show some results of the solutions, \( \gamma(z) \), with respect to different distances between two connectors \( (z_e = 2, 3, 3.41, 4, 4.66, \text{and } 6) \) and given \( \Delta \gamma_c = 0.5 \) and \( \beta = 0.5 \). In Fig. 2, the vertical coordinate represents the dimensionless radial displacement \( \gamma \), whereas the horizontal coordinate is the dimensionless distance from the \( i \)th connector \( (z - z_i) \). From Fig. 2, we see that the middle point between two adjacent connectors starts to contact the wall of the wellbore when \( z_i = 3.41 \) and wrap contact occurs when \( z_i > 4.66, \text{given } \Delta \gamma_c = 0.5 \text{ and } \beta = 0.5 \).

**No Contact.** When \( z_i < z_{i+1} \), the body of pipe between two adjacent connectors will not contact the wall of the wellbore. In this case, Eq. 11 is used to determine the configuration of a pipe in the interval of \( (z_i, z_{i+1}) \), and integral constants \( A_1 \) and \( A_2 \) are

\[
A_1 = \frac{\lambda_{c,1}}{2\beta^2}, A_2 = \frac{\lambda_{c,1}}{2\beta^2} \cot(2\lambda_{c,1}), \text{ where } \lambda_{c,1} = \sqrt{\frac{\beta z_{c,1}}{2\sqrt{2}}} \ldots (13)
\]

In case of no contact, it is easy to prove that \( \gamma(z) \), \( \gamma'(z) \), \( \gamma''(z) \), and \( \gamma'''(z) \) are continuous and smooth within the interval of \( (z_i, z_{i+1}) \), \( \gamma(z) \), \( \gamma'(z) \), and \( \gamma''(z) \) are also continuous at a connector \( (z_i \text{ or } z_{i+1}) \), however, \( \gamma'''(z) \) becomes discontinuous at \( z_i \) or \( z_{i+1} \) because of the action of a concentrated normal contact force that is applied at a connector (see Appendix C). From Eq. C-8, we can determine the dimensionless concentrated normal contact force at a connector,

\[
n_c = \frac{N_c}{W_L} = [\gamma'''(z_{i+1}) - \gamma'''(z_i)]/\lambda_{c,1} = 1, \ldots (14)
\]

or, equivalently, \( N_c = wL_c \).

The radial displacement, \( \gamma(z) \), reaches its maximum at the middle point between two connectors, \( z_m = z_i + 0.5z_e \), and \( \gamma'''(z_m) > 0 \). As shown by the thick solid black \( (z_e = 2) \) and thin solid black \( (z_e = 3) \) curves in Fig. 2, radial displacement, \( \gamma(z_m) \), increases as \( z_e \) increases. Let \( z_{i+1} \) denote the first critical distance between two connectors beyond which the middle point will contact the wall of the wellbore, \( z_{i+1} \) can be solved from \( \gamma(z_{i+1}) + 0.5z_e = 1 \), or equivalently,

\[
0.5\sqrt{2\lambda_{c,1}[\tan(\lambda_{c,1} - \lambda_{c,1})] = \eta, \ldots (15)\]

The solid black curve in Fig. 3 shows the plot of \( \lambda_{c,1} \) solved from Eq. 15.
Point Contact. When \( \zeta_i \geq \zeta_i,1 \), the middle point \( \zeta_{i,m} = \zeta_i + 0.5\zeta_c \) will contact the wall of the wellbore, as shown by the thick dashed black (\( \zeta_i = 3.41 \)) and thin dashed black (\( \zeta_i = 4 \)) curves in Fig. 2. In this case, Eqs. 11 and 12 are, respectively, used for the interval of \( (\zeta_i + 0.5\zeta_c, \zeta_{i,1}) \) and \( (\zeta_{i,1} - 0.5\zeta_c, \zeta_{i,1}) \). Integral constants \( A_1 \) and \( A_2 \) are

\[
A_1 = \frac{\lambda_n}{4\beta^2} + \frac{\Delta\zeta_i}{2\tan\lambda_n - 2\alpha}, \quad A_2 = \frac{\lambda_n}{4\beta^2} + \frac{\Delta\zeta_i}{2\tan\lambda_n - 2\alpha},
\]

Similarly, we obtain from Eq. C-8, \( n_c = \frac{\Delta\zeta_i}{\beta}\lambda_n \), [i.e., \( n_c + n_a = \frac{\Delta\zeta_i}{\beta}\lambda_n \) or \( N_c + N_p = \lambda_n \)]. The distributive normal contact force in the interval of \( (\zeta_i + 0.5\zeta_c, \zeta_{i,1}) \) is \( n = \frac{\lambda_n}{\beta} = 1 \) or \( N = w \). The preceding solutions clearly indicate that the concentrated contact forces acting at a connector and at the two endpoints of continuous contact \( (\zeta_i + 0.5\zeta_c, \zeta_{i,1} - 0.5\zeta_c) \) balance the weight of the pipe segment that does not contact with the wall of wellbore, whereas the weight of the pipe segment that continuously contacts the wall of the wellbore is balanced by the linearly distributive normal contact force, \( N = w \).

Stability Analysis of a Pipe With Connectors in Horizontal Wells

Numerical Solution of the Buckling Equation Using the Fourth-Order Runge-Kutta Method. When the axial compressive force is larger than the critical load of sinusoidal buckling (\( \beta > \beta_{cr} \)), the pipe’s axis becomes a 3D snaking-shaped curve [i.e., \( \theta(\zeta) \neq 0 \)]. Determination of the critical force, \( \beta_{cr} \), beyond which the pipe will change its configuration from a 2D curve to a 3D snaking shape, is crucially important for practical applications.

The solution of Eq. 10, \( \gamma(\zeta) \), represents the configuration of an unbuckled pipe with connectors and has been solved in the preceding. As discussed above, \( \gamma(\zeta) \) is a periodic function, but \( \gamma'(\zeta) \) is discontinuous at a series of points (at a connector or at a contact point). Thus, it is difficult to obtain an analytical solution of the linearized buckling equation with respect to the angular displacement (Eq. 9), and we need to seek numerical approaches. In this paper, we apply the fourth-order Runge-Kutta method, RK4 (Butcher 2003), to solve the linear differential equation.

Let \( y = (y_1, y_2, y_3, y_4)^T \) and Eq. 9 becomes

\[
\frac{dy}{d\zeta} = f(y, \zeta) = (f_1, f_2, f_3, f_4)^T, \quad \text{where} \quad y_1 = \theta(\zeta), \quad f_1 = \frac{d\theta}{d\zeta}, \quad y_2 = \frac{dy}{d\zeta}, \quad f_2 = \frac{dy}{d\zeta}, \quad y_3 = \frac{d^2\theta}{d\zeta^2}, \quad y_4 = \frac{d^2\gamma}{d\zeta^2}, \quad \text{and} \quad f_3 = \frac{d^2\gamma}{d\zeta^2}.
\]

The dashed black curve in Fig. 3 is the plot of the second critical distance, \( \lambda_{c,2}(\eta) \), solved from Eq. 17. The two curves, \( \lambda_{c,1}(\eta) \) and \( \lambda_{c,2}(\eta) \), divided the 2D plane of \( (\eta, \lambda_c) \) into 3 regions—\( A(\lambda_c < \lambda_{c,1}), B(\lambda_{c,1} < \lambda_c < \lambda_{c,2}), \) and \( C(\lambda_c > \lambda_{c,2}) \), which correspond to the three different configurations of an unbuckled pipe: no contact, point contact, and wrap contact, respectively. Given physical properties \( (w \text{ and } EI) \), geometrical parameters \( (r_w, p_w, F_w, \text{ and } L_o) \), and axial force \( (F_1) \), it is simple to calculate both \( \lambda_{c,2} \) and \( h \) and thus to decide the configuration of the pipe from the location of the point \( (\eta, \lambda_{c,2}) \) in Fig. 3.

Wrap Contact. When \( \zeta_i > \zeta_{i,2} \), part of the pipe in the middle part between two adjacent connectors continues contacting the wall of the wellbore, as shown by the thick dotted (\( \zeta_i = 4.66 \)) and thin dotted (\( \zeta_i = 6 \)) curves in Fig. 2. In this case, Eqs. 11 and 12 are, respectively, used for the interval of \( (\zeta_{i,1} + \zeta_{i,2}, \zeta_{i,2}) \) and \( (\zeta_{i,2} - \zeta_{i,1}) \). Integral constants \( A_1 \) and \( A_2 \) are

\[
A_1 = \frac{\lambda_n}{4\beta^2} + \frac{\Delta\zeta_i}{2\tan\lambda_n - 2\alpha}, \quad A_2 = \frac{\lambda_n}{4\beta^2} + \frac{\Delta\zeta_i}{2\tan\lambda_n - 2\alpha},
\]

Similarly, we obtain from Eq. C-8, \( n_c = \frac{\Delta\zeta_i}{\beta}\lambda_n \), [i.e., \( n_c + n_a = \frac{\Delta\zeta_i}{\beta}\lambda_n \) or \( N_c + N_p = \lambda_n \)]. The distributive normal contact force in the interval of \( (\zeta_{i,1} + 0.5\zeta_c, \zeta_{i,2}) \) is \( n = \frac{\lambda_n}{\beta} = 1 \) or \( N = w \). The preceding solutions clearly indicate that the concentrated contact forces acting at a connector and at the two endpoints of continuous contact \( (\zeta_{i,1} + 0.5\zeta_c, \zeta_{i,1} - 0.5\zeta_c) \) balance the weight of the pipe segment that does not contact with the wall of wellbore, whereas the weight of the pipe segment that continuously contacts the wall of the wellbore is balanced by the linearly distributive normal contact force, \( N = w \).

\[\theta(\zeta_i) = \theta_0 (\zeta_i - \zeta_{i,2}) \pm 0.5 \theta_{cr}(\zeta_i - \zeta_{i,2}) \quad \text{for} \quad \zeta_i > \zeta_{i,2} \]

The error per step of the RK4 method is on the order of \( h^5 \).
solution for any arbitrary constant \( c \neq 0 \) if \( \theta'(\zeta) \) is a solution of the equation. Changing the value of \( \theta'(0) \) is equivalent to changing the value of the constant \( c \). Our purpose is to find a nontrivial solution that satisfies the boundary conditions. Thus, we can fix \( \theta'(0) \) to any nonzero value—for example, \( \theta'(0) = 0.1 \)—in our implementation. Now our problem becomes finding a \( \theta''(0) \) such that both \( \theta(z_L) = 0 \) and \( \theta'(z_L) = 0 \) hold. Let us define an objective function

\[
F_1(\theta''(0), \beta) = \theta^2(z_L) + [\theta'(z_L)]^2 + \{[\theta'(z_L)]^2 - [\theta'(0)]^2\}. \tag{22}
\]

As shown in Fig. 4, \( F_1(\theta''(0), \beta) \) has a unique minimum, \( F_{\text{min}}(\beta) \), when \( \beta \) is fixed. Given different values of \( \beta \) (0.7, 0.8, and 0.9, as shown in Fig. 4), the minimum is different. The minimum can be solved easily by applying a 1D line-search optimization method. Fig. 5 illustrates the plot of \( F_{\text{min}}(\beta) \) for a given setting of parameters (\( \Delta \gamma_L = 0.5 \) and \( \zeta_c = 2.5 \)). We need to note that \( F_{\text{min}}(\beta) \) has zero solutions at a series of discrete values of \( \beta \).

A nontrivial solution \([\theta(\zeta) \neq 0]\) that satisfies boundary conditions on both ends exists when \( \beta \) equals these values. The smallest value of such \( \beta \) for which \( F_{\text{min}}(\beta) = 0 \) holds is the critical force of buckling.

Similarly, we can obtain both \( \beta_{\text{cr}} \) and \( \theta''(0) \) for a pipe with a pinned end at \( \zeta = 0 \) and a fixed end at \( \zeta = z_L \) by minimizing the objective function of

\[
F_2(\theta''(0), \beta) = \theta^2(z_L) + [\theta'(z_L)]^2 \tag{23}
\]

for fixed \( \beta \) and by solving \( F_{\text{2min}}(\beta) = 0 \).

For a pipe with two fixed ends, the boundary conditions of \( \theta(0) = 0 \) and \( \theta'(0) = 0 \) are given, and \( \theta''(z_L) = \pm \theta''(0) \) holds because of symmetry. We fix \( \theta''(0) = 0 \) and then determine both \( \beta_{\text{cr}} \) and \( \theta''(0) \) by minimizing the objective function of

\[
F_3(\theta''(0), \beta) = \theta^2(z_L) + [\theta'(z_L)]^2 + \{[\theta'(z_L)]^2 - [\theta'(0)]^2\}, \tag{24}
\]

for fixed \( \beta \) and by solving \( F_{\text{3min}}(\beta) = 0 \).

**Numerical Results and Discussion.** Figs. 6 through 9 show some numerical results obtained by applying the numerical method just discussed. In these examples, a pipe is pinned at both ends. In Figs. 6, 7, and 9, the vertical coordinate represents dimensionless critical force. In Fig. 8, the vertical coordinate is dimensionless wavelength. In Figs. 6, 7, and 8, the horizontal coordinate is dimensionless length of a pipe. In Fig. 9, the horizontal coordinate is dimensionless distance between two adjacent connectors.

Gao et al. (1994) derived analytical solutions for a pipe without connectors (\( \Delta \gamma_L = 0 \)); also see Gao and Miska (2009a):

\[
\beta_{\text{cr}}(z_L, \zeta_c, 0) = \frac{1}{2} \left( \frac{p_{\text{cr}}}{p_{\text{cr}}} + 1 \right) = \frac{1}{2} \left[ \left( \frac{k_\pi}{z_L} \right)^2 + \left( \frac{z_L}{k_\pi} \right)^2 \right]. \tag{25}
\]
where integer \( k \geq 1 \) is the number of half-waves and is chosen such that it minimizes \( \beta_{cr}(\zeta_c, \zeta_c, 0) \). It is obvious that the critical force is independent of \( \zeta_c \) (the distance between two adjacent connectors) when \( \Delta \gamma_c = 0 \). The wavelength is \( \lambda_c = \frac{2\pi k}{C_0} \). If \( \Delta \gamma_c = k\pi \), then \( \Delta \gamma_c = 2\pi, \beta_{cr}(k\pi, \zeta_c, 0) = 1 \). For a long pipe, as \( \zeta_c \to \infty, \Delta \gamma_c \to 2\pi \) and \( \beta_{cr}(\zeta_c, \Delta \gamma_c, 0) \to \beta_{cr}(\infty, \zeta_c, 0) = 1 \). The solid curve in Fig. 6 is the critical force calculated using the analytical solution of Eq. 25.

The validity of the numerical method presented in this paper can be examined by comparing numerical results with analytical results for the case of a pipe without connector (\( \Delta \gamma = 0 \)). Open circles in Fig. 6 are obtained with the numerical method. As shown in Fig. 6, numerical results are identical to analytical solutions. The fluctuation of critical force is caused by a change of wave length for discrete values of \( k \).

Our numerical results (Fig. 7) show that both the distance between two adjacent connectors (\( \zeta_c \)) and radius difference between a connector and the pipe body (\( \Delta \gamma \)) have significant impact on the critical force. Results shown in Fig. 7 are obtained by setting \( \Delta \gamma = 0.25 \). Different symbols in Fig. 7 are numerical results calculated with respect to different distances between two adjacent connectors, \( \zeta_c = 0.5, 1, 1.5, 2, 2.5, 3, 3.5, \) and 4.

When the distance between two connectors is small (\( \zeta_c \leq 1.5 \)), radial deflection is rather small and its effect on critical force is negligible; thus, the radial displacement can be regarded as a constant of \( \gamma_c = 1 - \Delta \gamma_c \). It is equivalent to a pipe with no connector being constrained in a wellbore with a reduced clearance of \( (1 - \Delta \gamma_c)r_0 \). When \( \zeta_c \leq 1.5 \), the critical force and wavelength can be approximated by

\[
\beta_{cr}(\zeta_c, \Delta \gamma_c) = \beta_{cr}(\infty, \zeta_c, 0) \sqrt{1 - \Delta \gamma_c} \tag{26}
\]

\[
\zeta_c = \frac{2\pi k}{(1 - \Delta \gamma_c)^{0.25}} \tag{27}
\]

The black solid curves in Figs. 7 and 8 are critical force and wavelength calculated by Eqs. 26 and 27 given \( \Delta \gamma_c = 0.25 \). Numerical results of critical force and wavelength with respect to \( \zeta_c = 0.5 \) (open black triangles), 1 (black stars), and 1.5 (open black circles) are almost identical to the solid black curve. The zigzag shape of the solid curve in Fig. 8 represents the change of the number of half-waves as the length of the pipe increases. For a short pipe with \( \zeta_c < \zeta_c < 2\sqrt{\pi} \), one complete wave (\( k = 2 \)) will be formed. Thus, the wavelength will be reduced from \( 2\sqrt{\pi} \alpha \) to \( \zeta_c \) at \( \zeta_c = \zeta_c \).

Numerical results shown in Figs. 7 and 8 clearly show that the effect of boundary conditions is negligible, and thus critical force and wavelength become independent of the pipe’s length for a pipe that is long enough (\( \zeta_c > 5\pi \)).

However, as the distance between two adjacent connectors increases, radial deflection becomes quite large (as shown in Fig. 2), and the effect of radial deflection on critical force and wavelength becomes significant. The approximate analytical solutions (Eqs. 26 and 27) no longer hold for \( \zeta_c > 1.5 \), and we have to apply the numerical method to determine the critical force and wavelength. Radial displacement \( y(z) \) increases as \( \zeta_c \) increases, and thus critical force decreases, as shown by the dashed curve with solid dots (\( \zeta_c = 2 \)), the dashed curve with solid triangles (\( \zeta_c = 2.5 \)), and the dashed curve with open diamonds (\( \zeta_c = 3 \)) in Fig. 7. The critical force reaches its minimum at \( \zeta_c = \zeta_c,1 \). When \( \zeta_c > \zeta_c,1 \), the middle point between two connectors contacts the wellbore, which provides a support or an extra constraint to the pipe, and thus makes it more stable. Further increasing the distance between two connectors may increase the critical force, as illustrated by the dashed black curve with open squares (\( \zeta_c = 3.5 \)) and the dashed black curve with solid diamonds (\( \zeta_c = 4 \)) in Fig. 7.

You may notice that some solid squares (\( \zeta_c = 2.5 \)) and solid dots (\( \zeta_c = 3.5 \)) are not on the zigzag black curve in Fig. 8. Our numerical results show that the wavelength remains constant, \( \zeta_u = 2\zeta_c \) (i.e., a half-wave is formed within two adjacent connectors in such cases). Within a certain range (\( -0.2\pi < \zeta_c,1 - k\pi < 0.2\pi \)), the dimensionless distance between two connectors is close enough to \( k\pi \), and half-waves are formed within two connectors, no matter how long the pipe is.

Let \( \beta_{cr,x}(\zeta_c, \Delta \gamma_c) = \beta_{cr}(\infty, \zeta_c, \Delta \gamma_c) \) denote the dimensionless critical force of a long pipe. \( \beta_{cr,x}(\zeta_c, \Delta \gamma_c) \) depends on both \( \zeta_c \) and \( \Delta \gamma_c \). Plots in different styles in Fig. 9 are numerical results of \( \beta_{cr,x}(\zeta_c, \Delta \gamma_c) \) with respect to \( \Delta \gamma_c = 0.1, 0.3, 0.5, \) and 0.7, respectively. These results clearly show that both \( \zeta_c \) and \( \Delta \gamma_c \) have significant impact on the critical force. The dashed black horizontal lines in Fig. 9 are analytical results predicted by Eq. 26. The black solid vertical line in Fig. 9 represents \( \zeta_c = 1.5 \). When \( \zeta_c \leq 1.5 \), critical force increases as \( \Delta \gamma_c \) increases, and numerical results are very close to analytical solutions predicted by Eq. 26 using \( \beta_{cr}(\infty, \zeta_c, 0) = 1 \) with respect to different values of \( \Delta \gamma_c \), which further supports the validity of the numerical algorithm presented in this paper.

Because the effect of radial deflection of pipe becomes dominant when \( \zeta_c > 1.5 \), critical force decreases dramatically as the distance between two connectors (\( \zeta_c \)) increases and reaches its minimum at \( \zeta_c = \zeta_c,1 \) (the critical distance of point contact). The larger the radius difference between a connector and pipe (\( \Delta \gamma_c \)), the more reduction of critical force is observed. Geometrical parameters and physical properties used to determine critical force of some commonly used drillpipes and sucker rods are listed in Table 1. For these commonly used pipes, the range of \( \Delta \gamma_c \) is from 0.4 to 0.8, the range of \( \zeta_c \) is from 2.3 to 3.3, and critical forces of sinusoidal buckling may reduce by 20 to 60%, as shown by black open circles in Fig. 9.

**Conclusions**

1. A pipe with connectors in horizontal wells assumes a 2D curve in the vertical plane \([0(z) = 0]\) when axial force is smaller than the critical force of buckling \( (\beta < \beta_{cr}) \), and it will change its configuration from a 2D curve to a 3D snaking curve \([0(z) \neq 0]\) when the axial force exceeds the critical force.

2. An unbuckled pipe with connectors may take three different configurations: no contact, point contact, and wrap contact. Critical conditions of transformation from no contact to point contact, or from point contact to wrap contact, depend on distance between two adjacent connectors (\( \zeta_c \)), difference of outer radius between pipe and connector (\( \Delta \gamma_c \)), and axial force (\( \beta \)), all in dimensionless form.

3. A numerical algorithm was proposed in this paper to determine the critical force by solving the buckling equations numerically using the RK4 method. The numerical algorithm is validated by comparing numerical results with analytical results for the specific case of a pipe with no connector.

4. Existence of a connector may increase or decrease the critical force of buckling, depending on both distance between two adjacent connectors (\( \zeta_c \)) and difference of outer radius between pipe and connector (\( \Delta \gamma_c \)).

5. When the distance between two adjacent connectors is small (\( \zeta_c < 1.5 \)), the effect of radial deflection on critical force is
**Distance between two centralizers in case of using centralizers.

5-1/2
3-1/2
4-1/2
5/8
¼

may reduce by 20 to 60%.

and the pipe body, m

pipe over the interval of wrap contact, N/m


Buckling and Bifurcation of Oil Well Tubular

References

Buckling and Bifurcation of Oil Well Tubular

**Distance between two centralizers in case of using centralizers.

Nomenclature

References


**Distance between two centralizers in case of using centralizers.

TABLE 1—CRITICAL FORCE FOR COMMONLY USED DRILLPIPES AND SUCKER RODS

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<tr>
<th>rW (mm)</th>
<th>rpo (mm)</th>
<th>rpr (mm)</th>
<th>r0 (mm)</th>
<th>Lc (m)</th>
<th>EI (KN·m²)</th>
<th>ΔF/c</th>
<th>z/c</th>
<th>β</th>
<th>rpo</th>
<th>rW</th>
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* Equivalent weight of sucker rod in an inclined well with inclination angle of θ.

** Distance between two centralizers in case of using centralizers.

negligible. Existence of a connector provides extra constraints on the pipe and thus increases the critical force by a factor of \((1 - \Delta r_{cc})^{0.5}\). However, the effect of radial deflection becomes more significant and the critical force decreases as the distance between two connectors increases, and it reaches its minimum when the distance approaches the critical distance of point contact. Depending on the magnitude of \(\Delta r_{cc}\), the critical force may reduce by 20 to 60%.

Nomenclature

\[ E = \text{Young's elastic modulus, N/m}^2 \]
\[ EI = \text{bending stiffness of the pipe, N·m}^2 \]
\[ F_L = \text{axial compressive force acting at the loading end, N} \]
\[ \vec{F} = \text{force vector, N} \]
\[ l_p = \text{length of the part of pipe near a connector that does not contact the wall of the wellbore, m} \]
\[ L = m L_c = \text{total length of a pipe, m} \]
\[ L_c = \text{distance between two adjacent connectors, m} \]
\[ m = \text{number of segments of pipe divided by connectors} \]
\[ M = \text{moment vector, m·N} \]
\[ M = \text{magnitudes of moment vector, m·N} \]
\[ n = \text{dimensionless normal contact force} \]
\[ \hat{r} = \text{unit vector in the normal direction} \]
\[ n_c = \frac{N_c}{r_c} = \text{dimensionless normal concentrated contact force acting on a connector} \]
\[ n_p = \frac{N_p}{r_p} = \text{dimensionless normal concentrated contact force acting at the middle point between two connectors in case of point contact} \]
\[ n_a = \frac{2 N_a}{r_a} = \text{dimensionless normal concentrated contact force acting at the starting point of contact in case of wrap contact} \]
\[ N = \text{distributive normal contact force per length of pipe over the interval of wrap contact, N/m} \]
\[ N_c = \text{normal concentrated contact force acting on a connector, N} \]
\[ N_o = \text{normal concentrated contact force acting at the starting point of contact in case of wrap contact, N} \]
\[ N_p = \text{normal concentrated contact force acting at the middle point between two connectors in case of point contact, N} \]
\[ r = \text{radial displacement, m} \]
\[ r_0 = r_w - r_{po} = \text{clearance between pipe and the wall of wellbore, m} \]
\[ r_{co} = \text{outer radius of a connector, m} \]
\[ \Delta r_c = r_{co} - r_{po} = \text{difference of outer radius between a connector and the pipe body, m} \]

Subscripts

\( b \) = caused by bending deformation
\( crs \) = critical value for sinusoidal buckling
\( p \) = pipe
\( r \) = radial direction
\( \theta \) = component along the angular direction

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Appendix A—Displacements and Deformation of a Pipe in an Inclined Well

Let \( x \) denote the distance of a point \( P(x, 0, 0) \) on the axis of an inclined wellbore from the origin \( O(0, 0, 0) \). \( i \) is a unit vector along the axis of the inclined wellbore, the direction of which is pointing from \( O \) to \( P \), \( j \) and \( k \) are unit vectors that are perpendicular to \( i \). Let \( r(x) \) and \( \theta(x) \) denote the radial and angular displacements at a point \( Q(x, y, z) \) on the axis of the pipe. The location of the point \( Q(x, y, z) \) can be determined by a vector, 

\[
\mathbf{r}_C = x^i + r \sin \theta^j - r \cos \theta^k \quad \text{.......................... (A-1)}
\]

The tangential vector of the pipe's axis is defined as 

\[
\mathbf{t} = \frac{dr}{dx} = \dot{i} + \frac{dr}{dx} \dot{\rho} + r \frac{d\theta}{dx} \dot{q} \quad \text{.......................... (A-2)}
\]

\[
\mathbf{\rho} = \sin \theta \mathbf{t} - \cos \theta \mathbf{k} \quad \text{...................(A-3)}
\]

\[
\mathbf{\theta} = \cos \theta \mathbf{t} + \sin \theta \mathbf{k} \quad \text{..................(A-4)}
\]

The normal and binormal vectors of the pipe's axis are, respectively,

\[
-k \mathbf{\kappa} = \frac{d^2 \mathbf{r}}{dx^2} = k_r \mathbf{\rho} + k_\theta \mathbf{\theta} \quad \text{.................. (A-5)}
\]

\[
k \mathbf{b} = \mathbf{t} \times (k \mathbf{k}) = \left(k_r \frac{d\theta}{dx} - k_\theta \frac{dr}{dx}\right) \mathbf{t} + k_\theta \mathbf{\rho} - k_r \mathbf{\kappa} \quad \text{.............. (A-6)}
\]

\[
k_r = \frac{d^2 r}{dx^2} - r \left(\frac{d\theta}{dx}\right)^2 \quad \text{.................. (A-7)}
\]

\[
k_\theta = 2 \frac{dr}{dx} \frac{d\theta}{dx} + \frac{d^2 \theta}{dx^2} \quad \text{.............. (A-8)}
\]

The first term in Eq. A-6 is negligible because it is in the second order of \( r \).

Appendix B—Derivation of Buckling Equations

With Application of Static Equilibrium Equations

Now, let us consider all forces and moments acting on the differential pipe element between the two normal planes at \( x \) and \( x + dx \). Force \( F(x) \) and moment \( M(x) \) are acting on one end \( (x) \) of the differential pipe element. Force \( \dot{F}(x + dx) \) and moment \( \dot{M}(x + dx) \) are acting on the other end \( (x + dx) \) of the differential pipe element.

In case of the pipe element contacting the wall of the wellbore, a normal contact force, \(-N \mathbf{d}x\), is applied to it. The distributive force per unit length of the pipe is

\[
\mathbf{F} = -w \cos \alpha \mathbf{i} - N \mathbf{p} - w \sin \alpha \mathbf{z} = f_x \mathbf{i} + f_r \mathbf{\rho} + f_\theta \mathbf{\theta} \quad \text{......... (B-1)}
\]

where \( \alpha \) denotes the inclination angle of the wellbore, \( w \) denotes the weight of pipe per unit length. Let \( f_x, f_r \), and \( f_\theta \) denote the distributive forces per unit length along \( i, \rho \), and \( \theta \), respectively,

\[
f_x = -w \cos \alpha, \quad f_r = -N + w \sin \alpha \cos \theta, \quad f_\theta = -w \sin \alpha \sin \theta \quad \text{.......................... (B-2)}
\]

From elastic-beam theory, the following relationship holds:

\[
\mathbf{M} = -E I \mathbf{b} \quad \text{.......................... (B-3)}
\]

Using Eq. A-6 in Eq. B-3 yields

\[
\mathbf{M} = M_r \mathbf{\rho} + M_\theta \mathbf{\theta} \quad \text{.......................... (B-4)}
\]
where
\[ M_p = -Elk_0 = -EI \left( 2 \frac{dr}{dx} \frac{d\theta}{dx} + \frac{d^2 \theta}{dx^2} \right) \]  
\hspace{1cm} \text{(B-5)}

and
\[ M_q = Elk_r = EI \left[ \frac{d^2 r}{dx^2} - r \left( \frac{d\theta}{dx} \right)^2 \right] \]  
\hspace{1cm} \text{(B-6)}

The derivative of \( \bar{M}(x) \) with respect to \( x \) is
\[ \frac{d\bar{M}}{dx} = \left( \frac{dM_p}{dx} - M_q \frac{d\theta}{dx} \right) \bar{p} + \left( \frac{dM_q}{dx} + M_p \frac{d\theta}{dx} \right) \bar{q}. \]  
\hspace{1cm} \text{(B-7)}

The differential pipe element remains in its static equilibrium state under the combined actions of \(-F\), \(-\bar{M}\), \(F\), \(x\), \(\bar{M}(x + dx)\), and \(f\). Thus, the following balance equations hold:
\[ \frac{dF}{dx} = -f \]  
\hspace{1cm} \text{(B-8)}

\[ \frac{d\bar{M}}{dx} = \bar{f} \times \bar{r}. \]  
\hspace{1cm} \text{(B-9)}

Decomposing vector \( F \) along the three orthogonal unit vectors of \( \bar{r}, \bar{p}, \) and \( \bar{q} \),
\[ F = F_x \bar{r} + F_y \bar{p} + F_z \bar{q}, \]  
\hspace{1cm} \text{(B-10)}

where \( F_x(x) \) is the axial force. \( F_x(x) > 0 \) means a tensile force, whereas \( F_x(x) < 0 \) means a compressive force. The derivative of \( F \) with respect to \( x \) is
\[ \frac{dF}{dx} = \left( F_x \frac{d\theta}{dx} - F_y \frac{dr}{dx} \right) \bar{r} + \left( F_y - F_x \frac{d\theta}{dx} \right) \bar{p} \]
\[ + \left( F_z \frac{dr}{dx} - F_x \right) \bar{q}. \]  
\hspace{1cm} \text{(B-11)}

From Eqs. B-10 and A-2, we have
\[ \frac{dF}{dx} = \left( F_x \frac{d\theta}{dx} - F_y \frac{dr}{dx} \right) \bar{r} + \left( F_y - F_x \frac{d\theta}{dx} \right) \bar{p} 
+ \left( F_z \frac{dr}{dx} - F_x \right) \bar{q}. \]  
\hspace{1cm} \text{(B-12)}

Using Eqs. B-5, B-6, B-7, and B-12 in Eq. B-9 gives
\[ F_q = F_s \frac{d\theta}{dx} + EIr \left[ \frac{d\theta}{dx} \frac{d^3 \theta}{dx^3} - \frac{d^3 \theta}{dx^3} - 3EI \frac{d^2 r}{dx^2} \frac{dr \frac{d^2 \theta}{dx^2} + \frac{d^2 \theta}{dx^2} \frac{dr \frac{d^2 \theta}{dx^2}}{dx} \right] \]  
\hspace{1cm} \text{(B-13)}

\[ F_p = F_s \frac{dr}{dx} - EIr \frac{d^3 r}{dx^3} + 3EI \left[ \frac{dr \frac{d^2 \theta}{dx^2} + \frac{d^2 \theta}{dx^2} \frac{dr \frac{d^2 \theta}{dx^2}}{dx} \right] \]  
\hspace{1cm} \text{(B-14)}

The term \( F_p \frac{d\theta}{dx} - F_q \frac{dr}{dx} \) in Eq. B-12 is negligible. Using Eqs. B-1, B-2, B-11, B-13, and B-14 in Eq. B-8 gives
\[ \frac{dF_s}{dx} = -w \cos \theta, \]  
\hspace{1cm} \text{(B-15)}

where \( F_s \) is the axial compressive force acting at the loading end \((x = L)\).

\[ N = -EIr \left[ \frac{d^4 \theta}{dx^4} - 3 \left( \frac{d^2 \theta}{dx^2} \right)^2 \right] \]
\[ -4 \frac{d^4 \theta}{dx^4} \frac{F_L - w(L - x) \cos \theta}{EI} \]
\[ - EIr \frac{d}{dx} \left( \frac{d^3 r}{dx^3} - 6 \frac{d^2 r}{dx^2} \frac{d^2 \theta}{dx^2} + \frac{d^3 \theta}{dx^3} \frac{dr}{dx} + \frac{F_L - w(L - x) \cos \theta}{EI} \frac{dr}{dx} \right) \]
\[ + \text{wsinzcos}\theta \]  
\hspace{1cm} \text{(B-16)}

\[ EIr \frac{d}{dx} \left( \frac{d^3 \theta}{dx^3} - 2 \frac{d^3 \theta}{dx^3} + \frac{F_L - w(L - x) \cos \theta}{EI} + \text{wsinzsin}\theta \right) \]
\[ + 4EI \left( \frac{d^3 \theta}{dx^3} - 3 \frac{d^3 \theta}{dx^3} \right) \frac{dr \frac{d^3 \theta}{dx^3}}{dx^2} \]
\[ + \frac{dr}{dx} \left( \frac{d^3 \theta}{dx^3} + \frac{F_L - w(L - x) \cos \theta}{2EI} \frac{dr}{dx} \right) \]
\hspace{1cm} \text{(B-17)}

**Appendix C—Action of Concentrated Normal Contact Force**

Let us consider the situation of applying a concentrated force on a pipe at a point \( x_p \). The concentrated force is
\[ F_c(x_p) = F_c(x_p) \bar{r} + F_c(x_p) \bar{p} + F_c(x_p) \bar{q}. \]  
\hspace{1cm} \text{(C-1)}

The differential pipe element at \( x_p \) remains in static equilibrium under combined action of \(-F_c(x_p), -\bar{M}(x_p), -\bar{M}(x_p), \) and \( F_c(x_p) \). Applying force-balance equations to the differential pipe element gives
\[ \bar{M}(x_p) = M(x_p) \]  
\hspace{1cm} \text{(C-2)}

\[ F_c(x_p) = F_c(x_p) \]  
\hspace{1cm} \text{(C-3)}

\[ F_c(x_p) = F_c(x_p) \]  
\hspace{1cm} \text{(C-4)}

and
\[ F_c(x_p) = F_c(x_p) \]  
\hspace{1cm} \text{(C-5)}

Bending moment \( M(x_p) \) is continuous at \( x_p \). However, the axial force and shear forces in both normal and binormal directions become discontinuous because of the action of the concentrated force.

Because a pipe is elastic, the pipe’s axis is still a smooth function at \( x_p \), and thus \( r, \theta, \frac{dr}{dx}, \frac{d\theta}{dx}, \) and \( \frac{dr}{dx} \) are continuous at \( x_p \). From Eqs. A-7, A-8, B-5, and B-6 and the continuous nature of bending moments at \( x_p \), we can infer that both \( \frac{dr}{dx} \) and \( \frac{d^2 \theta}{dx^2} \) are continuous at \( x_p \). Using Eqs. B-13 and C-3 in Eq. C-5 gives
\[ \left( \frac{d^4 \theta}{dx^4} \right)_{x_p} = \frac{F_c(x_p)}{EI} - \frac{F_c(x_p) \frac{d\theta}{dx}}{EI} \]  
\hspace{1cm} \text{(C-6)}

Similarly, using Eqs. B-14 and C-3 in Eq. C-4 yields
\[ \left( \frac{d^3 \theta}{dx^3} \right)_{x_p} = \frac{F_c(x_p)}{EI} - \frac{F_c(x_p) \frac{d\theta}{dx}}{EI} \]  
\hspace{1cm} \text{(C-7)}

Eqs. C-6 and C-7 indicate that the concentrated normal contact force acting at a point makes both \( \frac{dr}{dx} \) and \( \frac{d^2 \theta}{dx^2} \) discontinuous at the same point. In general, both axial and tangential components, \( F_c(x_p) \) and \( F_c(x_p) \), of a concentrated force acting at a point
When axial compressive force is smaller than the critical force of sinusoidal buckling ($\beta < \beta_c$), the pipe’s axis will remain in a vertical plane [i.e., $\theta(\xi) \equiv 0$ holds]. The equilibrium configuration of an unbuckled pipe can be solved from the following equation that is normalized and simplified from the buckling equation of Eq. B-16 by assuming $\theta(\xi) \equiv 0$ and $n(\xi) \equiv 0$:

$$\frac{d^4y}{dx^4} + 2\beta \frac{d^2y}{dx^2} - 1 = 0. \quad \text{(D-1)}$$

As discussed in Appendix C, the shear force and the third derivative $y'''(\xi)$ are discontinuous at a connector ($\xi_i = \xi_c$) or at a contact point ($\xi_c = \xi_c + \Delta \xi_c$) where a concentrated normal contact force is applied. The solution of Eq. D-1 can be represented by a piecewise continuous function. The general solutions of Eq. D-1 in the interval of $(\xi_i, \xi_i + \Delta \xi_i)$, $\gamma_{i1}(\xi_i)$, can be represented as

$$\gamma_{i1}(\xi) = A_1 \sin[\sqrt{2\beta}(\xi - \xi_i)] + A_2 \cos[\sqrt{2\beta}(\xi - \xi_i)] + \frac{1}{2\beta} \sqrt{\Delta \xi_i} \xi_i + A_4, \quad \text{(D-2)}$$

where $A_j$ ($j = 1, 2, 3, 4$) are integral constants that can be determined from boundary conditions at $\xi_i$ and $\xi_i$. Because the length of a connector is much shorter than the distance between two adjacent connectors and the bending stiffness of a connector is much greater than the bending stiffness of the pipe’s body, it is realistic to simplify the problem by assuming that a connector is a rigid disk with zero length. The boundary conditions at the $i$th connector are

$$\gamma_{i1}(\xi_i) = 1 - \Delta \gamma_c, \quad \frac{d^3y_{i1}}{dx^3}|_{\xi_i} = 0, \quad \text{(D-3)}$$

where $\Delta \gamma_c = \frac{\pi \Delta c}{r_0}$ is the dimensionless difference between the outer radius of a connector and the outer radius of the pipe.

Using Eq. D-2 in Eq. D-3 gives

$$A_4 = 1 - \Delta \gamma_c, \quad A_2, A_3 = -\sqrt{2\beta} A_1, \quad \text{ (D-4)}$$

Eq. D-2 becomes

$$\gamma_{i1}(\xi) = A_1 \{\sin[\sqrt{2\beta}(\xi - \xi_i)] - \sqrt{2\beta}(\xi - \xi_i)\} + A_2 \{\cos[\sqrt{2\beta}(\xi - \xi_i)] - 1\} + \frac{1}{2\beta} (\xi - \xi_i)^2 + 1 - \Delta \gamma_c \quad \text{(D-5)}$$

When the distance between two adjacent connectors is small ($\xi_i < \xi_i$), no contact exists between the body of the pipe and the wall of the wellbore except at connectors. The integral constants $A_1$ and $A_2$ can be solved from the boundary conditions at $\xi_{i+1}$:

$$\gamma_{i1}(\xi_{i+1}) = 1 - \Delta \gamma_c, \quad \frac{d^3y_{i1}}{dx^3}|_{\xi_{i+1}} = 0 \quad \text{(D-6)}$$

$$A_1 = \frac{x_0}{2\beta}, \quad A_2 = \frac{x_0 \cot(\gamma_c)}{2\beta^2} \quad \text{(D-7)}$$

Where $\lambda_c = \sqrt{\beta c}$. When the body of the pipe contacts the wall of the wellbore at a point $\xi_p = \xi_c + \Delta \xi_p$, the integral constants can be solved from the contact conditions at $\xi_p$:

$$\gamma_{i1}(\xi_p) = 1, \quad \frac{d^3y_{i1}}{dx^3}|_{\xi_p} = 0 \quad \text{(D-8)}$$

$$A_1 = \frac{\lambda_p}{2\beta}, \quad A_2 = \frac{\lambda_p \cot \lambda_p}{2\beta^2} + \frac{\Delta \gamma_c}{2\lambda_p \cot \lambda_c - 2} \quad \text{(D-9)}$$

In case of point contact, the contact point is at the middle between two adjacent connectors, $\Delta \xi_p = 0.5 \xi_i$, and $\lambda_p = \lambda_c = \sqrt{\beta c}$. Whereas the contact point is at $\xi_p = \xi_c + \xi_a$, $\Delta \xi_p = \xi_a$ and $\lambda_p = \lambda_a = \sqrt{\beta c}$ in case of linear (or wrap) contact.

Similarly, we can also derive the solution of Eq. D-1 in the interval of $(\xi_i + \Delta \xi_i, \xi_i + \Delta \xi_i + 1)$, $\gamma_{i2}(\xi)$:

$$\gamma_{i2}(\xi) = A_1 \{\sin[\sqrt{2\beta}(\xi + \xi_i) - \xi_c] - \sqrt{2\beta}(\xi + \xi_i) - \xi_c\} + A_2 \{\cos[\sqrt{2\beta}(\xi + \xi_i) - \xi_c] - 1\} + \frac{1}{2\beta} (\xi + \xi_i - \xi)^2 + 1 - \Delta \gamma_c \quad \text{(D-10)}$$

### SI Metric Conversion Factors

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*Conversion factor is exact.

Guohua Gao works for Shell as a senior reservoir engineer. His research interests include production optimization, automatic history matching, reservoir management, and mechanics of tubulars. He holds BS and MS degrees in mechanical engineering and a PhD degree in petroleum engineering. He serves as an Associate Editor for SPE Journal and as a reviewer for different SPE journals.

Qifeng Di is a professor at Shanghai University. His research interests include mechanics of tubulars, wellbore hydraulics, managed-pressure drilling, and directional drilling.

Wenchang Wang is a postdoctoral student at Shanghai University. His research interests include tubular and engineering mechanics. He holds a BS degree in solid mechanics and MS and PhD degrees in engineering mechanics.