Dynamic Well Intervention Modeling
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Abstract
A dynamic soft-string finite-element torque and drag model for modeling coiled tubing, wireline and drilling operations in a wellbore has been developed. The model includes the ability to simulate axial and torsional stick-slip for complex dynamic situations such as stall torque, jarring and yo-yoing. This paper documents the equations used to develop this model, compares model results with analytical equations and presents several example simulations performed by the model.

Introduction
Types of Models
There are many numerical models available for modeling downhole forces and torques for wireline (WL) (which includes slickline, electric line and braided cable), coiled-tubing (CT) and drill-string (DS). The generic term “string” will be used to refer to a conveyance conduit composed of any of these 3 conveyance methods. These models are often known as torque and drag models, or as tubing forces models, or sometimes well intervention models. Most of these models are “soft-string”, which means the bending stiffness of the pipe is ignored. Some are “stiff-string” which means the bending stiffness is included in the calculation. Stiff-string modeling is typically much more complex than soft-string modeling. Except in special situations, the bending stiffness of the pipe in the well does not significantly affect the forces in the pipe, and thus the less sophisticated soft-string models are sufficient for most applications. Most soft-string models do contain special calculations for specific stiff-string phenomena such as buckling (and the associated wall contact forces) and bottom-hole assembly (BHA) or tool bending. The special situations in which stiff-string models are required include running pipe in a hole with small clearances (e.g., casing in a torturous open hole), or running around severe bends (e.g., pushing through an elbow in a pipeline).

Most of the models available on the market (both soft-string and stiff string) are “steady-state” models which means they do not include dynamic forces in their calculations. Steady-state models allow the string to be moving at a certain axial speed and possibly rotating at a certain constant RPM. But they do not consider the acceleration forces which occur when there is a change in speed (axial and/or rotational). Of the models that do perform dynamic simulations (References 2-10), few include the “stick-slip” phenomena. Stick-slip occurs when a portion of the string stops moving for a short period of time. Once it stops, the force required to overcome static friction is greater than the force required to overcome dynamic friction. When the string begins to move there will be a sudden jerk as the friction force changes from static to dynamic friction. Thus there is often a repeated sticking and slipping instead of smooth, continuous motion.

Dynamic Modeling
Dynamic models are typically developed for a particular situation such as BHA analysis, drill-string vibration and whirl, and jarring simulations. They may be either soft-string or stiff-string and they may use an analytical solution or a time-transient solution. This paper discusses a dynamic, soft-string, time-transient model which includes stick-slip. This model is being developed for well-intervention and drilling applications. This model is being used to simulate the following types of dynamic situations:

- Yoyo – variations in tool/BHA depth due to the yo-yoing of the string which may be damped or exacerbated by stick-slip
- Bit stick – sudden sticking of the bit when a DS is being rotated may cause the DS to twist-off
- Bit stick-slip – a bit may stick, then release, then stick again causing torsional vibrations
• Working tension down – in WL applications additional tension can be worked down to the tools by up and down motion at surface.

• Jarring – with and without an accelerator.

The dynamic forces are composed of acceleration forces (force = mass times acceleration) and damping forces (force = damping times velocity). The acceleration forces are obviously important. According to reference 2, damping is also important in this type of modeling, though it is difficult to know how much damping to use.

**Model Equations**

The model described here, named Athena, is a soft-string, dynamic model which incorporates stick-slip. It uses one dimensional finite-element model of the string for the axial forces and displacements, and for the rotational torques and twists. It uses finite-difference techniques to include the dynamic effects.

**Finite-Element Model**

The same finite-element (FE) technique was used for both the force and the torsional model. The model, described in Figure 1, divides the string into “elements”. There is a “node” at the end of each element. Two adjacent elements share a common node, so if there are n elements in a string there are n+1 nodes. Half of the mass of each element is lumped at the node at each end of the element. A spring constant k, is defined for each element. The axial spring constant, k\textsubscript{ax}, and rotational spring constant, k\textsubscript{rot} are:

\[
\begin{align*}
  k_{\text{ax}} &= \frac{AE}{L} \\
  k_{\text{rot}} &= \frac{GJ}{L}
\end{align*}
\]  

The spring constant, k, multiplied by the difference in the displacements for the 2 nodes is the force in the element, r. The force in the element is constant, with forces only changing at the nodes. Equation (2.1) are the two basic equations that can be written for one element. These equations can be written in matrix form as shown in Equation (2.2). Matrix Equation (2.3) show how these equations can be combined for 3 elements. The r values cancel out except for the applied forces at the top and bottom of the string. The combined K matrix is a tri-diagonal matrix. A large spring constant, K\textsubscript{hold} is applied at the top or bottom of the string, to hold the string from moving.

All of the forces or torques applied to the string are added to the R matrix. These forces are calculated analytically and added to each node. The forces included in the R matrix are the weight and friction forces, and any other applied force such as those caused by fluid drag or a differential pressure. The friction forces are caused by wall contact forces (WCF). The following things cause WCF:

• Weight – the component of the weight vector normal to the wellbore causes WCF

• Capstan or belt affect – the WCF is a function of the tension or compression in the string around a curve

• Buckling – sinusoidal or helical buckling cause additional WCF

• Wellbore Tortuosity – the small helical nature of many wellbores may cause additional WCF

• Tool or BHA bending – although this is a soft-string model, additional WCF due to bending of the BHA (bottom hole assembly) are included. This is of primary importance for WL applications.

Several of these WCF’s depend upon the axial force in the string. Thus, Athena runs the FE model to determine the axial forces, then recalculates the WCF and updates the R matrix, and reruns the FE model. These iterations are repeated until the solution converges. If the string is rotating, the rotation FE model is also run during these iterations.

**Finite-Difference Schemes**

The dynamic forces are included in Athena by using a finite-difference (FD) scheme. Consider the matrix equation which includes the dynamic forces:

\[
MA_j + BV_j + KU_j = R_j
\]  

Assume j is the current time step for which the calculation is being performed. j-1 is the previous time step, j-2 the time step before that, etc. The following FD schemes allow matrix Equation (1.2) to be rewritten as:

\[
\left[ K_{\text{dyn}} + K \right]U_j = R_j + R_{\text{dyn}}
\]  

Once the dynamic components are incorporated into the K and R matrices, the FE models can be run with the dynamic forces included.
**Simple FD Equations**

A simple FD scheme for time steps of \( dt \) seconds can be used for the velocity and acceleration values as follows:

\[
V_j = \frac{U_j - U_{j-1}}{dt} \tag{1.4}
\]
\[
A_j = \frac{U_j - 2U_{j-1} + U_{j-2}}{dt^2}
\]

Substituting the above equations into Equation (1.2) and rearranging yields:

\[
K_{\text{dyn}} = \frac{M}{dt} + \frac{B}{dt} \tag{1.5}
\]
\[
R_{\text{dyn}} = \frac{M}{dt^2} \left[ 2U_{j-1} - U_{j-2} \right] + \frac{B}{dt} \frac{U_{j-1}}{dt}
\]

There are times (discussed in the jarring portion of this paper) when it is desirable to vary the time steps during a simulation. Defining \( dt_j \) as the most recent time step and \( dt_{j-1} \) as the time step before that, the above equations can be rewritten as:

\[
\bar{dt} = \frac{1}{2} \left( dt_j + dt_{j-1} \right)
\]
\[
K_{\text{dyn}} = \frac{M}{dt_j dt_{j-1}} + \frac{B}{dt_{j-1}} \tag{1.6}
\]
\[
R_{\text{dyn}} = \frac{M}{dt_j} \left[ \frac{U_{j-1}}{dt_{j-1}} + \frac{U_{j-1}}{dt_{j-1}} - \frac{U_{j-2}}{dt_{j-1}} \right] + \frac{B}{dt_{j-1}} \frac{U_{j-1}}{dt_{j-1}}
\]

This simple FD scheme uses a velocity that is actually half a time step behind the current time and an acceleration that is a full time step behind the current time (Equation (1.4)). It has the advantage that it is based on only the last two time steps and thus it responds quickly to sudden changes in velocity and acceleration.

**Forward Difference FD Scheme**

A more complex FD scheme can be used in which the velocity and acceleration are extrapolated forward to the current time. The forward difference velocity and acceleration equations are:

\[
V_j = \frac{\left( 3U_j - 4U_{j-1} + U_{j-2} \right)}{2dt} \tag{1.7}
\]
\[
A_j = \frac{2U_j - 5U_{j-1} + 4U_{j-2} - U_{j-3}}{dt^2}
\]

In the same manner, the dynamic components of the K and R matrix can be calculated.

\[
K_{\text{dyn}} = \frac{2M}{dt^2} + \frac{3B}{2dt} \tag{1.8}
\]
\[
R_{\text{dyn}} = \frac{M}{dt^2} \left[ 5U_{j-1} - 4U_{j-2} + U_{j-3} \right] + \frac{B}{2dt} \left[ 4U_{j-1} - U_{j-2} \right]
\]

In a similar manner to the simple FD scheme, these equations can be rewritten for a variable time step. This forward difference FD scheme is more accurate for gradually changing situations because the velocity and acceleration are approximated for the current time. However, it requires 3 previous time steps, and thus is not as responsive to sudden changes in velocity and acceleration.

Both of these FD schemes are available in Athena.

**Stick-Slip**

Steady-state models tend to use dynamic friction coefficients because they assume the string is moving at a constant velocity. Stick-slip is included in Athena by considering the static friction coefficient to be some “stick-slip factor” (specified by the user) times the dynamic friction coefficient. So if a stick-slip factor of 1.1 is used, the static friction is 10% greater than the dynamic friction.

To determine if a node is stuck, Athena uses the velocity calculated using Equation (1.4). When the velocity becomes less than a user specified “sticking velocity”, Athena assumes the node sticks. A node is able to slip again when the force applied to the node by the adjacent elements is large enough to overcome the static friction force.
Experimental data or field data is needed to help determine realistic values of the stick-slip factor and the sticking velocity.

**Stress Waves**

This section of the paper attempts to explain briefly how stress waves propagate through a string. Theoretical equations are used to calculate the speed, amplitude and reflected amplitudes of these waves. None of these equations are actually in Athena. Results from Athena are compared to the theoretical results to demonstrate the accuracy of Athena and aid in the understanding of these wave propagation phenomena.

**Propagation Speed**

When a change in axial-force occurs in a string, an axial tension or compression stress wave travels through the string. The sound speed at which this axial tension or compression wave propagates through the string material is:

\[ C_c = \sqrt{\frac{E g}{\rho}} \]  

(1.9)

Likewise when a change in torque occurs in a string, a torsional shear stress wave propagates through the string. These shear waves propagates at a slower speed given by:

\[ C_s = \sqrt{\frac{G g}{\rho}} \]  

(1.10)

For a typical steel material with a modulus of elasticity, E of 30,000,000 psi and Poisson’s ratio of 0.3, the axial tension/compression wave speed is 16,866 ft/sec and the torsional or shear wave speed is 10,460 ft/sec.

Cables (except for slickline) are more complicated because they are a composite structure. The speed of a tension wave through a cable can be approximated by converting the typical cable properties of diameter, weight and stretch coefficient, to an equivalent density and modulus of elasticity. Once these are known, the speed of a tension wave can be approximated. Table 1 gives these values for some typical cables.

### Table 1 - Compression Wave Speeds for Typical Cables

<table>
<thead>
<tr>
<th>Cable</th>
<th>Equivalent</th>
<th>Axial Wave Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diameter</td>
<td>Weight</td>
<td>Stretch</td>
</tr>
<tr>
<td>in</td>
<td>lb/ft</td>
<td>ft/Kft/Klb</td>
</tr>
<tr>
<td>0.185</td>
<td>0.066</td>
<td>3.070</td>
</tr>
<tr>
<td>0.224</td>
<td>0.096</td>
<td>2.200</td>
</tr>
<tr>
<td>0.288</td>
<td>0.157</td>
<td>1.600</td>
</tr>
<tr>
<td>0.322</td>
<td>0.195</td>
<td>1.200</td>
</tr>
<tr>
<td>0.380</td>
<td>0.269</td>
<td>1.000</td>
</tr>
<tr>
<td>0.426</td>
<td>0.316</td>
<td>0.750</td>
</tr>
<tr>
<td>0.464</td>
<td>0.321</td>
<td>0.770</td>
</tr>
<tr>
<td>0.521</td>
<td>0.470</td>
<td>0.580</td>
</tr>
</tbody>
</table>

**Wave Amplitude**

There is an amplitude of force change associated with a tension or compression wave which depends on the excitation velocity, \( V_e \), which causes the wave. The force change is:

\[ \Delta F = V_e C_c \frac{W}{g} \]  

(1.11)

Likewise a torsional wave has an associated change in torque which depends on the excitation speed, \( \text{RPM}_e \) which causes the wave. The torque change is:

\[ \Delta T = \omega_e C_s I_r \frac{W}{g} \]  

(1.12)

where \( \omega_e \) is the excitation rotational velocity and \( I_r \) is the rotational inertia:

\[ \omega_e = \text{RPM}_e \frac{2\pi}{60} \]  

(1.13)
As an example, consider a vertical well with the 0.464" cable from Table 1 running to a stuck tool located at 11,414 ft, which is the distance an axial wave will travel in 1 sec. The top of the cable was pulled up at a velocity of 300 ft/min (60 ft/sec). According to Equation (1.11), the change in force was 572 lb (using consistent units). This situation was simulated with Athena, and the results are shown in Figure 2:

- The increase in force at surface was 572 lb for the first second while the initial wave traveled to the bottom.
- The force at surface was then constant (even though the pick-up velocity remained constant) for 1 second while the initial wave propagated upward.
- The force at the bottom was unchanged during the first second while the wave propagated down.
- When the wave reached the bottom it reflected, causing the force at the bottom to increase by $2\Delta F$, or 1,144 lb.
- When the wave reached the surface again, beginning at 2 seconds, and reflected downward, the surface force also increased by $2\Delta F$.
- Each successive time the wave reflected off of the top or bottom, the force increased by $2\Delta F$.

For another example, consider a vertical well with a steel pipe 4” OD, 3.5” ID (no upsets) down to 10,460 ft, which is the distance a shear wave will travel in 1 second. According to Equation (1.14), $I_r$ is 0.4688 in$^2$ or 0.003255 ft$^2$. This pipe was turning freely at 100 RPM, which according to Equation (1.13) was a rotational velocity, $\omega$, of 10.47 rad/sec. At time 0, the bit (at the bottom of the pipe) suddenly stuck and no longer rotated. The torque, according to Equation (1.12), at the bit should change suddenly to 834 ft lb. This situation was simulated with Athena. The results are shown in Figure 3:

- The torque increased by 834 ft lb immediately at the bit and was then constant until 2 seconds while the shear wave traveled to the surface and back.
- The torque wave of 834 ft lb required 1 second to travel to the surface where it was reflected downward. The change in torque at the surface was doubled to 1,668 ft lb, once for the wave which traveled up and again for the reflected wave downward.

**Wave Reflections**

As was noted above, these waves reflect off of the top and bottom of the string. A portion of the wave is also reflected when there is a change in the properties of the string\(^{(l)}\). The amplitude of the reflected wave depends on the cross-sectional areas, densities and the modulus of elasticity for axial waves, and on the rotational mass moments of inertia and the shear modulus of elasticity for torsional waves. The wave reflection coefficient, $\lambda_c$, for axial compression and tension waves for the interface between two string components a and b is:

$$\alpha_c = \frac{A_a \sqrt{E_a \rho_a}}{A_b \sqrt{E_b \rho_b}}$$

and

$$\lambda_c = \frac{\alpha_c - 1}{\alpha_c + 1} \quad (1.16)$$

Likewise for the shear wave the reflection coefficient, $\lambda_s$, is given by:

$$\alpha_s = \frac{A_a I_m \sqrt{E_a \rho_a}}{A_b I_m \sqrt{E_b \rho_b}}$$

and

$$\lambda_s = \frac{\alpha_s - 1}{\alpha_s + 1} \quad (1.18)$$

As an example, consider steel pipe which was hanging in a vertical well to 16,866 ft. In the top half of the well the pipe OD was 6” and the ID was 5.25”. In the bottom half of the well the pipe OD was 4” and the ID was 3.5”. The bit on the end was stuck with initially no force at the bit. Starting at time 0 the pipe was picked up at surface at a velocity of 200 ft/min. The resulting simulation from Athena is shown in Figure 4. From Equation (1.12), the force change, $\Delta F$, in the top section of the pipe was 39,292 lb. The reflection coefficient, $\lambda_c$, was 0.3846:
Surface
- The surface force increased immediately by 39,292 lb.
- The wave traveled down to the interface between the two pipe sections at the middle of the string and a wave of $\Delta F$ multiplied by $\lambda_c$ or 15,112 lb, was reflected upward.
- When the reflected wave reached the surface (at 1 second) it reduced the surface force by double its amplitude or 30,224 lb.
- At 2 seconds the surface force increased again by double $\Delta F$, or 78,583 lb
- Thus the force at surface continued to increase by $(78,583 - 30,224) = 48,358$ lb for each 2 second period.

Bottom
- The portion of the wave that traveled to the bottom was $\Delta F$ multiplied by $(1 - \lambda_c)$ or 24,179 lb. At 1 second the force at the bottom increased from 0 to 24,179 lb.
- After this increase, the force at the bottom continued to increase by double this reflected amount or 48,358 lb.

Center
- The force at the center of the string increased at 0.5 seconds by the amplitude of the wave reflected downward or by 24,179 lb.
- This increase happened each second, so the rate of increase was 48,358 lb for every 2 second period.

Thus the force increased at the same average rate along the entire length of the string.

Stall Torque
So called “twist-off” failures are far too common in the drilling industry\(^\text{10}\). Consider another example in which a DS was being used to drill a 10,000 ft vertical well with 12 PPG drilling fluid. The upper 6,000 ft of the DS was 5.5” DP and the lower 4,000 ft was 2 7/8” DP. The API make up torque (MUT) of the 2 7/8” DP is 7,170 ft lb, so the torque limiter on the top drive was set at 7,000 ft lb. The DS was rotating at 100 RPM with 1,000 ft lb torque at bit when suddenly the bit stuck and remained stuck. Figure 5 shows the results from Athena for this situation:
- The torque at the bit increases immediately by 322 ft lb, which can be calculated with Equation (1.12)
- After nearly 1 second the torque at the surface begins to increase
- The torsional waves cause a undulation in the surface torque as it climbs toward the top drive limit
- When the top drive limit is reached, the top drive holds a constant torque of 7,000 ft lb at surface even when turning backwards
- The torque at the transition between the two sizes of DP at 6,000 ft is the same as the torque at the bit after the initial transients.
- The torque in the 2 7/8” DP reaches a maximum of 8,212 ft lb well above the MUT for this DP.

In a situation like this, when the MUT was exceeded, a twist-off could easily have occurred. The top drive limit could have been set lower to avoid this possible twist-off.

Now consider the same situation except the well was L-shaped, turning from vertical to horizontal between 5,500 and 6,500 ft with a 0.25 dynamic friction coefficient. In this case initial torque while drilling was 2,265 ft lb and the top drive limit was again set to 7,000 ft lb. The results of the Athena simulation are shown in Figure 6. The torque at 6,000 ft, at the cross-over between the two DP sizes, reached 8,000 ft lb, which was well above the MUT of the 2 7/8” DP.

Yo-yoing
Consider a 770 lb tool string on a 0.464” cable from Table 1 at 10,000’ vertical well. The cable was moved upward at surface 20 ft in 5 seconds. The tool string stuck at first until a force of 1,000 lb was reached, then it broke free. The results of the Athena simulation are shown in Figure 7. As would be expected, the tool string oscillated up and down after being released. The amplitude of the oscillation was approximately twice the stretch that was caused in the cable by the 1,000 lb perturbation force. One period of the first mode of oscillation required the tension wave to travel up and down the well 4 times. Thus the period of the oscillation for a cable without a downhole tool was:

$$P_{w/o\_tool} = \frac{4\cdot L}{C_c}$$  \hspace{1cm} (1.19)
When the tool was included, the period could be found from the following equations derived from section 10.8 of reference 11. First, iteratively solve for β where:

\[ \beta \tan \beta = \frac{WL}{W_T} \]  

(1.20)

Then the period could be calculated:

\[ P_{w_/\; tool} = 2\pi \frac{L}{\beta C_c} \]  

(1.21)

If there was no tool, \( W_T = 0 \), and for the first mode \( \beta = \pi/2 \). In this case Equation (1.21) became Equation (1.19). Note in Figure 7 that the force at the top of the cablehead approached zero, but never went into compression. Although Athena doesn’t use any of these equations, the results agreed with them.

Now consider the same situation in a well that was slanted at 45 degrees, without including stick-slip. The simulation results are shown in Figure 8. The oscillations were damped out quickly due to friction with the wellbore. According to reference 11 section 3-10, coulomb friction does not change the period of oscillation, which agreed with the Athena results. Again, the force at the top of the cablehead narrowly avoided going into compression.

Finally, consider the slanted case again with stick-slip included. The dynamic friction coefficient was 0.25 and the static friction coefficient was 0.30. The simulation results are shown in Figure 9. Again, the oscillations were dampened out quickly. In this case the cable did go into compression at the cablehead. If the cable were rigid the compressive load would have reached approximately 400 lb. Since the cable was not rigid, it buckled in the wellbore.

**Working Tension Down**

Consider the 0.464” wireline in a 16,000’ S shaped well, with the tool string stuck at bottom. The cable was picked up 100 ft at surface, lowered 25 ft, then picked back up to the 100 ft position, lowered 6.25 ft, and picked back up to the 100 ft position. This up and down motion at surface with decreasing amplitude worked additional tension down to the stuck tools. A simulation of this situation is shown in Figure 10 along with a small isometric plot of the well shape. For this simulation a dynamic friction coefficient of 0.25 and a static friction coefficient of 0.30 were used. The velocity below which the cable would stick was 0.01 ft/min.

- During the initial pickup to 100 ft, the tension at the bottom increased to 5,500 lb.
- Slacking off at surface and then picking up again allowed additional tension to work down, until the tension at the tools reached a maximum of 7,150 lb.

Unfortunately the results from this type of stick-slip simulation vary with several parameters including the static and dynamic friction coefficients, the sticking velocity and even the FD time step. More work must be done to calibrate these results with actual measured results.

**Jarring**

Jarring requires a much more complicated dynamic analysis, primarily because the time required for the compression waves travel up and down the string is on the order of 1 second, while the time required for the compression waves to travel through the tool string and fish is on the order of 0.003 sec. When analyzing what happens in the string, a time step of 0.01 sec is typically used. However, when analyzing what happens in the tool string, a time step of 0.001 is needed. Thus Athena uses variable time steps in the FD scheme. Because of its complexity, jarring will be the subject of a future paper. One simple example is presented here.

A 0.464” cable from Table 1 was used in a 10,000 ft vertical well. The 3.5” tool string containing a jar was connected to a 220 lb fish. The fish was attached at the bottom to the formation with a 200,000 lb/in spring constant. The mechanical jar travel distance was 3.5” and the upward release force was 4,000 lb. The cable was moved upward at surface at a rate of 300 ft/min starting at time zero. A time step of 0.001 sec was used except for one second starting just before the jar released. During that one second a time step of 0.0001 was used. Figure 11 and Figure 12 show results from this simulation.

- The jarring caused compression waves to travel up and down through the tools and fish with a period of about 0.011 sec. This oscillation was most obvious in the force at the bottom of the fish, shown in orange in Figure 11. The upward tension force at the bottom of the fish reached 60,000 lb. However the oscillations due to the spring connection to the formation at the bottom of the fish and the compression waves in the tool and fish caused downward compressive forces on the fish which exceeded 20,000 lb.
- The force at the jar anvil reached as high as 65,000 lb. However, the force in the cable at the cablehead did not exceed 5,200 lb. The wave reflection coefficient, \( \lambda_c \), between the cable and the tool was 0.984, which means that less than 2% of the compression wave in the tool was being transmitted into the cable. The rest was being reflected.
In Figure 12, the jar displacement exceeded 3.5” due to the stretch of the tool, fish and spring holding the fish to the formation. The jar hammer bounced off of and on to the anvil due to the motions of the tool and fish. Variations in the stiffness of different tools in the tool string, the addition of an accelerator to the tools, allowing the fish to stick-slip, and damping all add to the complexity of a jarring simulation.

**Conclusion**

A dynamic, soft-string, finite-element model has been developed which enables many types of dynamic calculations to be performed. Results from this model have been compared to analytical solutions when possible. The stick-slip capability of this model may be novel in the industry, but has to be validated with field data. Jarring simulations are one of the most complicated dynamic simulations, and will be the subject of a future paper.

**Nomenclature**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>cross-sectional area</td>
</tr>
<tr>
<td>$A_j$</td>
<td>acceleration at time step $j$ (in/sec$^2$)</td>
</tr>
<tr>
<td>$A_a$</td>
<td>cross-sectional area of the $a$ section of the string (in$^2$)</td>
</tr>
<tr>
<td>$A_b$</td>
<td>cross-sectional area of the $b$ section of the string (in$^2$)</td>
</tr>
<tr>
<td>$C$</td>
<td>tension or compression speed of sound (in/sec or ft/sec)</td>
</tr>
<tr>
<td>$C_s$</td>
<td>torsional or shear speed (in/sec or ft/sec)</td>
</tr>
<tr>
<td>$E$</td>
<td>modulus of elasticity (psi)</td>
</tr>
<tr>
<td>$E_a$</td>
<td>modulus of elasticity for section $a$ of the string (psi)</td>
</tr>
<tr>
<td>$E_b$</td>
<td>modulus of elasticity for section $b$ of the string (psi)</td>
</tr>
<tr>
<td>$\Delta F$</td>
<td>change in force associated with a compression or tension wave (lb)</td>
</tr>
<tr>
<td>$g$</td>
<td>acceleration due to gravity (386.4 in/sec$^2$ or 32.2 ft/sec$^2$)</td>
</tr>
<tr>
<td>$G$</td>
<td>shear modulus (psi)</td>
</tr>
<tr>
<td>$I$</td>
<td>rotational moment of inertia (in$^2$)</td>
</tr>
<tr>
<td>$I_a$</td>
<td>rotational moment of inertia for section $a$ of the string (in$^2$)</td>
</tr>
<tr>
<td>$I_b$</td>
<td>rotational moment of inertia for section $b$ of the string (in$^2$)</td>
</tr>
<tr>
<td>$J$</td>
<td>polar moment of inertia (in$^4$)</td>
</tr>
<tr>
<td>$j$</td>
<td>the most recent of a series of time steps in the finite difference scheme</td>
</tr>
<tr>
<td>$k$</td>
<td>spring constant for an element</td>
</tr>
<tr>
<td>$k_{ax}$</td>
<td>axial spring constant (lb/in)</td>
</tr>
<tr>
<td>$k_{rot}$</td>
<td>rotational spring constant (in lb/rad)</td>
</tr>
<tr>
<td>$K$</td>
<td>matrix of spring constants used by the finite-element model</td>
</tr>
<tr>
<td>$K_{dyn}$</td>
<td>dynamic component of the $K$ matrix</td>
</tr>
<tr>
<td>$L$</td>
<td>length of the string (ft)</td>
</tr>
<tr>
<td>$M$</td>
<td>mass matrix of the finite-element model</td>
</tr>
<tr>
<td>$P$</td>
<td>period of oscillation of the downhole tool (sec)</td>
</tr>
<tr>
<td>$r$</td>
<td>force or torque in an element</td>
</tr>
<tr>
<td>$R$</td>
<td>matrix of applied forces or torques used by the finite element model</td>
</tr>
<tr>
<td>$R_{dyn}$</td>
<td>dynamic component of the $R$ matrix</td>
</tr>
<tr>
<td>$\Delta T$</td>
<td>change in torque associated with a torsional shear wave (ft-lb)</td>
</tr>
<tr>
<td>$u$</td>
<td>displacement of a node (in)</td>
</tr>
<tr>
<td>$U$</td>
<td>matrix of displacements used by the finite element analysis</td>
</tr>
<tr>
<td>$V_e$</td>
<td>excitation axial velocity (in/sec or ft/sec)</td>
</tr>
<tr>
<td>$V_j$</td>
<td>velocity at time step $j$ (in/sec)</td>
</tr>
<tr>
<td>$W$</td>
<td>weight per unit length of the string (lb/ft)</td>
</tr>
<tr>
<td>$W_T$</td>
<td>weight of the tool on the end of the string (lb)</td>
</tr>
<tr>
<td>$a_s$</td>
<td>ratio of section and material properties for compression wave reflection</td>
</tr>
<tr>
<td>$a_s$</td>
<td>ratio of section and material properties for shear wave reflection</td>
</tr>
<tr>
<td>$\lambda_c$</td>
<td>reflection coefficient for compression wave</td>
</tr>
<tr>
<td>$\lambda_s$</td>
<td>reflection coefficient for shear wave</td>
</tr>
<tr>
<td>$\rho$</td>
<td>density (lb/in$^3$)</td>
</tr>
<tr>
<td>$\rho_a$</td>
<td>density of the $a$ section of the string (lb/in$^3$)</td>
</tr>
<tr>
<td>$\rho_b$</td>
<td>density of the $b$ section of the string (lb/in$^3$)</td>
</tr>
<tr>
<td>$w_e$</td>
<td>excitation rotational velocity (rad/sec)</td>
</tr>
</tbody>
</table>

**References**

For a single element the spring equations can be written as:

\[ k_i (u_{i+1} - u_i) = r_i \]
\[ k_i (u_i - u_{i+1}) = r_{i+1} = -r_i \]  

(2.1)

Which can be written matrix form as:

\[
KU = R
\]

\[
K = \begin{bmatrix}
    k & -k \\
    -k & k
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
    u_i \\
    u_{i+1}
\end{bmatrix}
\]

\[
R = \begin{bmatrix}
    r_i \\
    r_{i+1}
\end{bmatrix}
\]

(2.2)

For a string of 3 elements the matrix equations are:

\[
\begin{bmatrix}
    k_1 & -k_1 & \\
    -k_1 & k_1 + K_2 & -k_2 \\
    -k_2 & k_2 + K_3 & -k_3 \\
    -k_3 & k_3 + k_{hold}
\end{bmatrix}
\begin{bmatrix}
    u_1 \\
    u_2 \\
    u_3 \\
    u_4
\end{bmatrix}
= \begin{bmatrix}
    r_1 \\
    0 \\
    0 \\
    r_4
\end{bmatrix}
\]

Figure 1 - Finite-Element Model Sketch and Equations

Figure 2 - Picking Up 0.464" Cable at 300 ft/min Stuck at Bottom in a Vertical Well at a Depth of 11,414 ft
Figure 3 - 4" X 3.5" Pipe rotating at 100 RPM in a 10,460' Vertical Well - Bit Sticks at time 0

Figure 4 - Wave Reflection Example in 2 Sizes of Pipe
Figure 5 - Stall Torque Example in 10,000 ft Vertical Well

Figure 6 - Stall Torque, 10,000 ft L-Shaped Well
Figure 7 - Yo-yo of Wireline in Vertical Well

Figure 8 - Yo-yo of Wireline in 45 deg Slant Well without Stick-Slip
Figure 9 - Yo-yo of Wireline in 45 deg Slant Well with Stick-Slip

Figure 10 - Working Tension Down a Wireline in a 16,000 ft S Shaped Well
Figure 11 - Openhole Wireline Jarring, 10,000’ Vertical Well – 0 to 4 seconds

Figure 12 - Openhole Wireline Jarring, 10,000’ Vertical Well – 3.2 to 3.7 seconds